Augmenting Batch Exchanges with Constant Function Market Makers

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• Two ideas in exchange design with newfound popularity
• How should we combine them?
• Goal: Map out design space (no dominant design)
Exchange Model

- Users trade $N$ divisible, fungible assets through *limit orders*
  - “Sell 1 unit of $\mathcal{X}$ for at least 2 units of $\mathcal{Y}$”
Two Exchange Design Innovations
Batch Exchanges

- Execute batches of trades, all at once
- Input: Set of limit orders

1. Compute Prices
2. Trade in batch at price quotients
   - Meaningless units
   - No pairwise matching

- “Clearing” if no debt

Pricing Engine

Theorem (Arrow and Debreu, 1954)
\[ \exists \text{ unique } \* \text{ equilibrium prices } \{p_A\} \text{ and allocations that clear the market.} \]
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<table>
<thead>
<tr>
<th>Transaction</th>
<th>Price Quotient</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell 10 USD for EUR</td>
<td>p&lt;sub&gt;USD&lt;/sub&gt; = 9</td>
<td>10 USD</td>
</tr>
<tr>
<td>Sell 9 EUR for JPY</td>
<td>p&lt;sub&gt;EUR&lt;/sub&gt; = 10</td>
<td>140 JPY</td>
</tr>
<tr>
<td>Sell 1350 JPY for USD</td>
<td>p&lt;sub&gt;JPY&lt;/sub&gt; = &lt;sup&gt;-1&lt;/sup&gt;15</td>
<td>135 USD</td>
</tr>
<tr>
<td>Sell 10000 USD for EUR</td>
<td>p&lt;sub&gt;EUR&lt;/sub&gt; = 1000</td>
<td>10000 USD</td>
</tr>
</tbody>
</table>

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Pricing Engine

\[
\begin{align*}
p_{\text{USD}} &= 9 \\
p_{\text{EUR}} &= 10 \\
p_{\text{JPY}} &= \frac{1}{15}
\end{align*}
\]

\[
\begin{array}{c}
\text{Sell 10 USD for EUR} \\
\text{min } \frac{9}{10} \text{ USD} \\
\text{Sell 9 EUR for JPY} \\
\text{min } \frac{140}{\text{EUR}} \text{ JPY} \\
\text{Sell 1350 JPY for USD} \\
\text{min } \frac{1}{135} \text{ JPY} \\
\text{Sell 10000 USD for EUR} \\
\text{min } \frac{1000}{\text{USD}} \text{ EUR}
\end{array}
\]

\[
\begin{align*}
\text{€9} &\iff $10 \\
$10 &\iff ¥1350 \\
¥1350 &\iff €9
\end{align*}
\]
Two Exchange Design Innovations

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Theorem (Arrow and Debreu, 1954)

∃ unique* equilibrium prices \( \{p_A\} \) and allocations that clear the market.
Key Properties of Batch Exchange Model

1. Uniform prices (unique!) bring economic benefits
   - Pareto-Optimal (for limit orders)

2. Requires computing Arrow-Debreu exchange market equilibria

- **Sell 10 USD for EUR**
  \[
  \min \frac{9}{10} \text{USD} \leq \frac{p_{\text{USD}}}{p_{\text{EUR}}} = \frac{9}{10} \checkmark
  \]

- **Sell 9 EUR for JPY**
  \[
  \min \frac{140}{135} \text{JPY} \leq \frac{p_{\text{EUR}}}{p_{\text{JPY}}} = \frac{150}{135} \checkmark
  \]

- **Sell 1350 JPY for USD**
  \[
  \min \frac{1}{135} \text{USD} \leq \frac{p_{\text{JPY}}}{p_{\text{USD}}} = \frac{1}{135} \checkmark
  \]

- **Sell 10000 USD for EUR**
  \[
  \min 1000 \text{USD} \nless \frac{p_{\text{USD}}}{p_{\text{EUR}}} = \frac{9}{10} \times
  \]

**Pricing Engine**

- $p_{\text{USD}} = 9$
- $p_{\text{EUR}} = 10$
- $p_{\text{JPY}} = \frac{1}{15}$

€9 ⇔ $10
$10 ⇔ ¥1350
¥1350 ⇔ €9
Two Exchange Design Innovations
Constant Function Market Makers

- CFMM maintains reserves and a trading function $f(\cdot)$
- Accepts trade from $(x, y)$ to $(x', y')$ if and only if $f(x, y) \leq f(x', y')$
- Why?
  - Market-makers add liquidity
  - Automated
  - Computational simplicity
Several projects combine batch exchanges with CFMMs, using different mechanisms
- Penumbra, CoWSwap, [Walther, 2021], [Canidio and Fritsch, 2023]

What are the tradeoffs for different mechanisms for integrating CFMMs into batch exchanges?
How can batch exchanges draw on passive liquidity?

- **Model:**
  - $N$ assets $\mathcal{X} \in \mathcal{A}$
  - 1 batch exchange
  - Many CFMMs, with different curves, reserves
  - Also outside world—other exchanges, other users, ...
How can batch exchanges draw on passive liquidity?

- Axiom 1: Asset conservation
- Axiom 2: Uniform Prices $\{p_x\}_{x \in \mathcal{A}}$
  - No trade from $x$ to $y$ gets a better rate than $\frac{p_x}{p_y}$. 

These are standard market design assumptions, leading to classic theory results on Arrow-Debreu market equilibria.
Augmenting Batch Exchanges with CFMMs

How can batch exchanges draw on passive liquidity?

- Axiom 1: Asset conservation

- Axiom 2: Uniform Prices \( \{p_\mathcal{X}\}_{\mathcal{X} \in \mathcal{A}} \)
  - No trade from \( \mathcal{X} \) to \( \mathcal{Y} \) gets a better rate than \( \frac{p_\mathcal{X}}{p_\mathcal{Y}} \).

- Axiom 3: Limit orders make best responses
  - A limit order trades \( \mathcal{X} \) to \( \mathcal{Y} \) at no worse than the market rate \( \frac{p_\mathcal{X}}{p_\mathcal{Y}} \), only if market rate exceeds limit price

- These are standard market design assumptions, lead to classic theory results on Arrow-Debreu market equilibria.
How can batch exchanges draw on passive liquidity?

- Axiom 4: CFMM trading function must not decrease
CFMMs in Batch Exchanges

How can batch exchanges draw on passive liquidity?

• Axiom 4: CFMM trading function must not decrease

Consequence
Market equilibrium is no longer unique
How can batch exchanges draw on passive liquidity?

- Axiom 4: CFMM trading function must not decrease

Consequence:
Market equilibrium is no longer unique

- How should a batch choose a CFMM’s trade?
- Also complicates equilibria computation
Asset Conservation and Uniform Prices imply:

**Consequence**
CFMMs must trade *at* market prices, not below
Some Desirable Properties

- **Pareto Optimality**
  - From perspective of limit orders
  - Recall: Without CFMMs, every equilibrium is Pareto Optimal

- **Price Coherence**
  - After a batch, CFMM spot exchange rates are quotients of some set of prices
  - Otherwise, cyclic arbitrage opportunity (free money)

- **Preservation of Price Coherence**
  - Price coherence, but only if prices are also coherent before batch
Consequences

Consequence

No mechanism can in all circumstances guarantee Pareto Optimality and (Preservation of) Price Coherence

- **Proof Intuition:**
  - PO can require trading all the way across
  - Multiple CFMMs with different curves will end at different spot exchange rates
Some More Desiderata and Consequences

- **Joint Price Discovery (JPD)**
  - After a batch, CFMM spot prices equal batch prices
  - Prevents a common atomic, risk-free “cyclic” arbitrage

- **JPD requires maximizing** $f(\cdot)$ (trading to $C$)

- **An example of how context matters:**
  - Trading to $C$ incentivizes splitting trade over many batches, but trading to $B$ does not.
  - How are batches initiated?
  - How many users?
Some More Desiderata and Consequences

- **Locally Computable Rule (LCR)**
  - CFMM trade depends only on CFMM state and market price

**Consequence**

Trading to C is a LCR that satisfies Price Coherence.
Some More Desiderata and Consequences

Consequence

Trading to $B$ is a LCR that guarantees Preservation of Price Coherence, if and only if all CFMMs use a constant product curve.

- **Unique** exception to incompatibility between PO and PPC
Computing Equilibria

- Mixed-Integer Programs [Walther21] or general (not always convex) solvers
- LCR ⇒ algorithms based on auctions, iterations (Tâtonnement) are directly applicable
  - LCR must satisfy Weak Gross Substitutability
  - Price goes up ⇒ demand does not increase
- What about other approaches? Convex programs?
A Convex Program for 2-Asset WGS Utility Functions

Observation
A CFMM trading between 2 assets, with a LCR satisfying WGS, acts like an (uncountably) infinite set of infinitesimal limit orders.

Let’s adapt a convex program for linear Arrow-Debreu exchange markets [DGV16] to support CFMMs trading between 2 assets.
Minimize \[ \sum_i p_i \left( e^i \ln \left( \frac{p_i}{\beta_i} \right) \right) - \sum_i y_{i,j} \ln u_{i,j} \]

Subject to \[ \sum_i y_{i,j} = \sum_i y_{j,i} \quad \forall j \in [N] \]
\[ p_j \geq 1 \quad \forall j \in [N] \]
\[ y_i \geq 0 \quad \forall i \in [M] \]
\[ u_{i,j} \beta_i \leq p_j \quad \forall i, j \]
A Convex Program for 2-asset WGS CFMM Trading Functions

Minimize \[ \sum_i p_{A_i} \int_0^\infty \left( d_i(z) \ln\left( \frac{p_{A_i}}{\beta_{i,z}(p)} \right) \right) dz - \sum_i p_{A_i} g_i(y_i/p_{A_i}) \]

Subject to \[ \sum_{i:A_i=j} y_i = \sum_{i:B_i=j} y_i \quad \forall j \in [N] \]
\[ p_j \geq 1 \quad \forall j \in [N] \]
\[ y_i \geq 0 \quad \forall i \in [M]. \]

Equivalently, this program solves exchange markets where each agent is interested in only two assets, using any WGS utility function on those two assets.
Conclusion

- **Axiomatic framework for integrating CFMMs into batch exchanges**
  - Extra degree of freedom requires deliberate choice
- **Natural desiderata are incompatible**
  - Pareto-Optimality at odds with Price Coherence
- **Convex program for exchange markets with 2-asset WGS CFMMs**