Batch Exchanges with Constant Function Market Makers: Axioms, Equilibria, and Computation

GEOFFREY RAMSEYER, Stanford University, USA
MOHAK GOYAL, Stanford University, USA
ASHISH GOEL, Stanford University, USA
DAVID MAZIÈRES, Stanford University, USA

Batch trading systems and constant function market makers (CFMMs) are two distinct market design innovations that have recently come to prominence as ways to address some of the shortcomings of decentralized trading systems. However, different deployments have chosen substantially different methods for integrating the two innovations.

We show here from a minimal set of axioms describing the beneficial properties of each innovation that there is in fact only one, unique method for integrating CFMMs into batch trading schemes that preserves all the beneficial properties of both. Deployment of a batch trading schemes trading many assets simultaneously requires a reliable algorithm for approximating equilibria in Arrow-Debreu exchange markets. We study this problem when batches contain limit orders and CFMMs. Specifically, we find that CFMM design affects the asymptotic complexity of the problem, give an easily-checkable criterion to validate that a user-submitted CFMM is computationally tractable in a batch, and give a convex program that computes equilibria on batches of limit orders and CFMMs. Equivalently, this convex program computes equilibria of Arrow-Debreu exchange markets when every agent’s demand response satisfies weak gross substitutability and every agent has utility for only two types of assets. This convex program has rational solutions when run on many (but not all) natural classes of widely-deployed CFMMs.
1 Introduction

A crucial component of an economic system is a structure to facilitate the exchange of assets. Typical exchanges facilitate trading of asset pairs through continuous double-auctions. Traders submit trade offers to an exchange, which either matches the new offer with an existing, compatible offer, or, if none exists, adds the new offer to its orderbooks. Each offer has a limit price, and will accept a trade that gives an exchange rate at or more favorable than the limit price.

One problem with continuous double auctions is that offers are processed in a serialized ordering. In practice, this means that whenever there is an arbitrage opportunity or an advantageous market fluctuation, the party that captures this value is the party whose trade offer reaches the exchange first. The result is continual capital investment in high-speed computer systems that, for those not participating in the arbitrage races, functions as a tax on liquidity provision [19]. The need to continually adjust offers is particularly problematic in blockchains with limited transaction throughput.

Budish et al. [23] propose using frequent batch auctions to address these challenges. Trades between two assets are accumulated over a short period of time, after which the exchange operator computes a uniform clearing price and settles as many trades as possible. Clearing every trade in a batch at the same price eliminates risk-free front-running opportunities [56]. The NYSE schedules trades in batches at the opening and closing of a trading day [9]. Batch trading schemes on blockchains have already been deployed [1] or are in development [7]; these batch schemes model a set of trades as an instance of an Arrow-Debreu exchange market [20], and clearing the batch requires (approximate) computation of an exchange market equilibrium.

Some batch trading schemes [1, 56] process a large number of assets in one batch, instead of just two assets, by computing in every batch a set of arbitrage-free exchange rates between every asset pair. This reduces the problem of liquidity fragmentation between different trading pairs, which is especially difficult on modern blockchains where liquidity is likely divided between widely-used blockchain-native assets such as Bitcoin and Ethereum, and between the multitude of USD-pegged tokens (so-called “stablecoins” [2, 3, 60]). Furthermore, this allows users to efficiently directly trade from any asset to any other, without ever holding some intermediate asset (such as USD). However, computation of an exchange market equilibrium is substantially more difficult when many assets are traded, than when there are just two.

Another recently-developed tool for improving exchange performance are Constant Function Market Makers (CFMMs), a class of automated market-makers [12]. Market-makers deposit capital into a CFMM, and the CFMM constantly offers trades according to a predefined trading strategy. This strategy is specified by some function of its asset reserves, and the CFMM accepts a trade if, after transferring assets in and out of its reserves, this function’s value does not decrease. Automation substantially reduces the number of blockchain transactions that a market-maker must execute. Chitra et al. [27] argue that the automatic nature of a CFMM’s response to trading grants robustness in case of a market shock or software outage. CFMMs like Uniswap [10] and Curve [34] are already some of the largest on-chain trading platforms.

Our focus is on how to combine these two market design innovations. Real-world deployments have already chosen fundamentally different methods for mediating the interaction, and making this choice correctly is crucial for developing an effective system that realizes the benefits of both designs simultaneously.

The core question is what trade should a CFMM make in response to a set of batch equilibrium prices. Figure 1 outlines some possibilities when trading between two assets. CoWSwap [1] uses rule B [61] and the Stellar blockchain plans to use rule C/D [45]. Angeris et al. [13] give an optimization framework for trading against a set of CFMMs that, (for a linear objective function) is equivalent to rule A.

1.1 Our Results

We give (§4) a minimal set of axioms describing batch trading schemes and CFMMs, and from these axioms show that rules C and D of Figure 1 are equivalent and are the unique, optimal way for combining CFMMs and batch trading while preserving the beneficial properties of both systems. All of the choices in Figure 1 have been deployed or seriously considered for deployment and have different real-world consequences, so getting this choice right is crucial.

This settles the question of which integration rule to use and has practical implications for real-world systems [1, 7]. These proofs are straightforward, and the main contribution is the minimal axiomatization that is well supported by practical motivations and leads to this unique, optimal choice. For example, one important property of batch trading systems is the elimination of arbitrage opportunities within a batch; this ultimately eliminates Rule B of Figure 1.

Furthermore, these axioms lead to a complete specification for the equilibrium computation problem in batch trading schemes that include CFMMs. This problem must be solved before any trading can be performed in an instance of a batch system. Specifically,
we show (§5) that a CFMM following the optimal trading rule can be modeled within the existing Arrow-Debreu exchange market framework that batch trading schemes already use as an agent using the CFMM trading function as its utility function.

Prior work of Codenotti et al. [29] gave a polynomial-time algorithm for this problem when agent demand functions satisfy a property known as “Weak Gross Substitutability” (WGS), but Chen et al. [25] show PPAD-hardness of the problem when agents display any so-called “non-monotonicity” (a property slightly stronger than non-WGS). We show that many natural classes of CFMM trading functions satisfy WGS and can be integrated directly into existing algorithms (§5), but that seemingly-natural trade requests and easy to specify trading functions map directly to the hard problem instances of [25]. Real-world deployments must therefore ensure that any user-submitted CFMM trading function satisfies WGS; we find a natural subclass of utility functions that admit a natural description format that is easy to validate and sufficient but not necessary to guarantee WGS.

This resolves the question of polynomial time computation, but formulation as a convex program has led to improved results in computational efficiency and structural understanding in other contexts (such as the much-celebrated result of Eisenberg and Gale [35]). We give a convex program (§6) for the problem of computing equilibria in batch exchanges that incorporate CFMMs.

This program may be of independent interest; an equivalent statement is that this program computes equilibria in Arrow-Debreu exchange markets where agent utilities can be any arbitrary quasi-concave function of two assets, subject to the constraint that each agent’s demand response satisfies WGS.

Our convex program is based on the program for linear exchange markets of Devanur et al. [31]. The proof is quite technical, but the intuition is easy to state: we develop a viewpoint from which CFMMs appear as an uncountable collection of infinitesimally-sized limit sell offers. Unlike the analyses in [31] based on Lagrange multipliers, we had to go back to earlier techniques and directly apply Kakutani’s fixed point theorem.

When the density of this infinite collection of limit offers is a rational linear function, this program has rational solutions, but some CFMMs, such as those based on the Logarithmic Market Scoring Rule [42], can force even small batches to only admit irrational solutions.

To wrap up the discussion, we discuss CFMM trading fees (§7) and some open problems raised in this work (§8.1). In particular, we observe some qualitative hints that CFMMs that trade between two assets (in exchange markets, when agents have utilities for only two types of goods) appear to be more algorithmically tractable than those that simultaneously trade three or more; an understanding of this difference, if there is any, would be a valuable addition to the literature on CFMMs.
Preliminaries

Every asset considered in this work is divisible, fungible, and freely disposable.

3.1 Constant Function Market Makers

A Constant Function Market Maker (CFMM) is an automated trading strategy parameterized by a trading function \( f \). At any point in time, the CFMM owns a collection of some assets (its "reserves"). We will denote CFMM reserves by \( x = \{x_a | a \in \mathcal{A} \} \) for some set of assets \( \mathcal{A} \) and each \( x_a \in \mathbb{R} \). All asset amounts are nonnegative. Reserves are provided by deposits from investors (so-called "liquidity providers").

The trading function determines whether a CFMM accepts a proposed trade. A CFMM with reserves \( x \) and function \( f \) accepts any trade that results in reserves \( x' \) (i.e. a trade of \( x' - x \)), so long as \( f(x') \geq f(x) \). In practice, rational traders trading with the CFMM only make trades that leave \( f(x') = f(x) \).

A standard assumption (e.g. [12]) is that the trading function is quasi-concave on the positive orthant. We assume also that all trading functions are nonnegative and nondecreasing (in every coordinate) on the positive orthant. We assume for clarity of exposition in this section and §4 that \( f \) is strictly quasi-concave and always differentiable (so the Definition 3.2 is always well-defined). This is for clarity of exposition in §4 – for example, Definition 3.2 could replace gradients with subgradients, but this would make the statements of our axioms in §4 less understandable. Theorem 5.1 and subsequent discussions do not require these assumptions.

At any set of reserves, the gradient of the trading function specifies the exchange rate that the CFMM would offer for a trade of marginal size.

**Definition 3.1** (Spot Valuation): The spot valuation of asset \( A \) in a CFMM with trading function \( f \) at reserves \( \hat{x} \) is \( \frac{\partial f}{\partial x_A}(\hat{x}) \).

**Definition 3.2** (Spot Exchange Rate): The spot exchange rate from asset \( A \) to asset \( B \) for a CFMM with trading function \( f \) at reserves \( \hat{x} \) is \( \frac{\partial f}{\partial x_A}(\hat{x}) / \frac{\partial f}{\partial x_B}(\hat{x}) \).

We list some examples of widely deployed trading functions below.

**Example 3.3** (CFMM Trading Functions):

- The most widely known DeFi CFMM, Uniswap (versions 1 and 2), uses the Constant Product Rule [10], which sets \( f(a,b) = ab \). The spot exchange rate from \( A \) to \( B \) is \( b/a \).
- The Constant Sum Rule uses the trading function \( f(a,b) = ra + b \), for a constant \( r \). The spot exchange rate is \( \frac{1}{r} A \) per \( B \).
- Uniswap version 3 [11] uses a piecewise function, dividing the range of prices into slices and using a separately-defined constant-product function on each slice.
- The weighted constant product rule [50] uses the trading function \( f(a,b) = a^{w_a} b^{w_b} \) for positive constants \( w_a, w_b \), which gives a spot exchange rate of \( \frac{w_b}{w_a} \).
- The DeFi platform Curve [34] uses a combination of a sum term, \( a + b \), and a product-like term, \( 1/(a+b) \).
- The Logarithmic Market Scoring Rule [42] corresponds to the trading function \( f(a,b) = -(e^{-a} + e^{-b}) + 2 \) [57], for a spot exchange rate of \( e^{b-a} \).
- Angeris et al. [15] build a CFMM where the value of its reserves replicate the price of a covered call option.
- The Minecraft [51] modification package [4] uses a CFMM with trading function \( f(a,b) = ae^b \), for a spot exchange rate of \( 1/a \).

3.2 Batch Trading

In a batch trading scheme, users submit trades to an exchange operator, which gathers trades into batches (typically with some temporal frequency, e.g. one batch per second) and then clears the batches using a uniform set of clearing prices. A set of clearing prices is a "valuation" on each asset, \( \{p_a\}_{a \in \mathcal{A}} \). Every trade from asset \( A \) to asset \( B \) occurs at an exchange rate of \( p_A/p_B \), the ratio of the asset valuations.

Batch trading schemes are modeled [56, 61] as an instance of an Arrow-Debreu exchange market [20]. In this model, a set of agents trades a set of assets, and an "auctioneer" acts as an intermediary. Each agent has some "endowment" of assets and a utility function.  

The core algorithmic challenge for implementing a batch trading exchange is the computation of equilibria in these exchange markets. By general Arrow-Debreu exchange market theory, all of the market instances that arise in this paper contain an equilibrium.

The computation can be done in a straightforward manner (binary search) when only two assets are traded in a batch, but requires more complicated algorithms when many assets are simultaneously traded in a batch. Efficient algorithms have been the subject...
of much research [29–33, 39, 43, 44], and runtime often depends on agent utility functions.

Real-world batch trading schemes [9, 56, 61] support some or all of the following types of orders. Note that offers might be in either direction on a trading pair (i.e. one might sell USD to purchase EUR, or sell EUR to purchase USD).

Example 3.4 (Limit Sell Offer): A limit sell offer is an offer to sell up to \( k \) units of a good \( A \) in exchange for as many units of good \( B \) as possible, subject to the constraint that the offer receive at least a limit price of \( r_0 \) units of \( B \) for each unit of \( A \) sold.

This corresponds to an Arrow-Debreu exchange market agent with utility \( u(a,b) = r_0 a + b \).

Example 3.5 (Limit Buy Offer): A limit buy offer is an offer to purchase up to \( k \) units of a good \( B \) by selling as few units of a good \( A \) as possible, subject to the constraint that the offer receive at most a limit price of \( r_0 \) units of \( B \) per each \( A \) sold.

This corresponds to an Arrow-Debreu exchange market agent with utility \( u(a,b) = r_0 a + \min(k,b) \).

Observe that the utility of the limit sell offer equals the trading function of the constant-sum CFMM.

4 An Axiomatic Approach to CFMMs in Batch Trading Schemes

Our contribution in this section is a minimal set of axioms that captures the economically useful properties of batch trading schemes and CFMMs and leads a full specification of the equilibrium computation problem for batch exchanges that incorporate CFMMs. We consider here batch exchanges that may contain multiple limit sell offers and may contain multiple CFMMs.

4.1 Batch Trading Properties

4.1.1 Uniform Valuations

The core fairness and efficiency properties of a batch trading scheme stem from the fact that all limit orders in a batch trade at the same arbitrage-free exchange rates.

Axiom 1: An equilibrium of a batch trading scheme has a shared market valuation \( \{p_A > 0\} \) for each asset \( A \in \mathcal{A} \). The exchange rate from \( A \) to \( B \) is defined as \( \frac{p_A}{p_B} \). Every asset transfer from \( A \) to \( B \) occurs at this exchange rate.

Limit orders in a batch trading system trade if and only if they are offered a price above their limit prices. When the batch exchange rate equals an offer’s limit price, the batch equilibrium specifies how much trading the offer performs (see [20]).

Batch trading schemes model themselves as instances of an Arrow-Debreu exchange market, a model that satisfies this axiom, and compute these valuations by computing exchange market equilibria. Standard theory of these exchange markets shows that under mild conditions (e.g. condition \( * \) of [31]), nonnegative valuations that clear the market (satisfying Axiom 2 below) exist and are unique, up to rescaling. All of the exchange markets considered in this work have unique, nonzero clearing valuations.

4.1.2 Asset Conservation

Trading systems must also not create or destroy any units of an asset.

Axiom 2: Let \( z_A \) be the amount of each asset \( A \in \mathcal{A} \) owned in aggregate by all participants in a batch prior to executing the batch of trades (for some computed equilibrium), and let \( z'_A \) be the total amount owned by all participants after the batch.

It must be the case that \( z_A = z'_A \).

In practice [13, 56, 61], Axiom 2 is something that must be exactly preserved, not just approximated (as in much of the theoretical literature, e.g. [21, 30]). Deployments might also charge fees on trades; we discuss fees in §7, implemented as a post-processing step, and assume outside of §7 that there are no fees.

4.2 Axioms: CFMMs

4.2.1 Nondecreasing Trade Function

We axiomatize CFMM behavior according to only the constraints that are explicitly verified in practice.

Axiom 3: A CFMM accepts a trade from reserves \( x \) to \( x' \) if and only if \( f(x) \leq f(x') \).

Per the “constant function market maker” name, a CFMM allows any trade from reserves \( x \) to reserves \( x' \) so long as \( f(x) = f(x') \). One might imagine encoding this strict equality condition as an axiom; however, real-world CFMM deployments only check the weaker condition that \( f(x') \geq f(x) \) [5], and assume that rational traders trading directly with the CFMM will choose trades that make \( f(x) = f(x') \).

The strict equality condition is actually too strong to allow CFMMs to usefully integrate with batch trading schemes (see Example A.1). Again, the assets considered in this work are freely disposable, so liquidity providers never strictly prefer to have less of an asset than more of it.
4.3 Axioms: CFMMs and Batch Trading

A batch trading scheme needs a specification for how a CFMM acts in a batch. Specifically, it needs a map that takes in a CFMM’s state (its reserves, and its trading function) and the state of the rest of the batch (a list of limit offers, other CFMMs, and the asset valuations in a batch) and outputs a trade for the CFMM to make.

The assumption that a CFMM’s trading function is strictly quasi-concave means that, given an initial set of reserves \( \hat{x} \) and a set of batch valuations \( \{p_A\} \), there is an injective map from any set of valuations \( \{q_A\} \) to a set of reserves \( \hat{x}' \) at which the CFMM’s spot valuations are to \( q \), and \( p \cdot \hat{x} = p \cdot \hat{x}' \). The spot valuations \( q \) therefore imply a unique trade of \( \hat{x}' - \hat{x} \) which satisfies Axiom 1. As such, it suffices for the trading rule to specify only a CFMM’s post-batch spot valuations.

**Definition 4.1 (CFMM Trading Rule):** A Trading Rule for CFMMs in a batch exchange is a map \( F(\{x_A\}_{A \in \mathcal{A}}, \{p_A\}_{A \in \mathcal{A}}, \Gamma) \rightarrow \{q_A\}_{A \in \mathcal{A}} \) for \( \{x_A\}_{A \in \mathcal{A}} \) the reserves of a CFMM, \( f \) its trading function, \( \{p_A\}_{A \in \mathcal{A}} \) a set of batch valuations, and \( \Gamma \) the set of any and all other information of every other agent in the batch. The output \( \{q_A\}_{A \in \mathcal{A}} \) is a set of spot valuations of the CFMM (which implies a specific CFMM trade).

4.3.1 CFMM Independence

Our next axiom models CFMMs as independent financial instruments. As motivation, note that in real-world CFMM deployments, users trade against one CFMM at a time, and market makers provide capital to individual CFMMs. Capital deployed to one CFMM is locked there and is not algorithmically transferred to another CFMM. A market maker that puts capital into a CFMM is exposed to risk in the price movement of the assets traded by that CFMM, with a specific risk profile that is particular to that CFMM trading function [15]. Market makers should therefore not have exposure to price movements of assets not traded by the CFMMs in which they participate, and should not have their own risk exposure be directly influenced by CFMMs in which they do not participate.

**Axiom 4:** A CFMM trading rule \( F \) can be specified as a function of only the CFMM’s reserves, the CFMM’s trading function, and the set of batch valuations.

A consequence of this axiom is that a CFMM can be described by a "demand function"; the trade of a CFMM can be computed as a function only of the CFMM’s state and the batch valuations. Agents in Arrow-Debreu exchange markets can be described in the same manner.

This axiom is algorithmically useful when implementing a batch trading scheme, because an algorithm can treat each CFMM separately. This simplifies the equilibrium computation problem, both for practical implementations and for designing the convex program in §6.

4.3.2 No Internal Arbitrage

Motivating our penultimate axiom is another important property of batch trading schemes. Batch trading schemes display no arbitrage opportunities that are contained entirely among the batch participants — equivalently, a batch trading scheme redistributes arbitrage profits within the batch to batch participants. This axiom captures the notion that a batch trading scheme should not leave behind arbitrage opportunities between CFMMs after executing a batch of trades. These opportunities would imply that batch participants overall are strictly worse off as compared to another batch outcome. Note that batch trading schemes can only eliminate arbitrage on trading activity within the batch, not between the batch and external markets.

**Axiom 5:** After executing a batch of trades, there must be a set of asset valuations \( \{q_A\}_{A \in \mathcal{A}} \) such that the spot exchange rate between any two assets \( A \) and \( B \) is \( q_A/q_B \) on every CFMM that trades between \( A \) and \( B \).

If every CFMM agrees on the exchange rate between every pair of assets, and these agreed-upon exchange rates are quotients of asset valuations, then there necessarily are no residual arbitrage opportunities between CFMMs participating in the batch exchange.

Residual arbitrage losses are not just a hypothetical concern. In a real-world deployment, any leftover arbitrage opportunities could be captured by arbitrageurs immediately after a batch executes. The reason is that batch trading schemes do not typically have exclusive access to a CFMM. For example, the batch exchange CoWSwap [1] operates as an Ethereum [63] smart contract that sometimes executes trades through a standalone CFMM implementation like Uniswap [10]. But when the batch exchange is not actively clearing a batch, other users may trade with the CFMM directly.

4.3.3 Align Spot and Batch Valuations

To motivate our final axiom, observe that while trade rules C and D of Figure 1 satisfy all of these axioms, so also does the pathological trading rule in which a CFMM makes no trades at all. A CFMM is a market-maker, and one of the important roles of
a market-maker is to facilitate asynchronous trading between users. All else equal, a CFMM facilitates more trading, and therefore is qualitatively a better market-maker, when it performs more trading rather than less.

We believe that Axiom 6 is a natural statement in its own right, and has an easy-to-state mathematical formulation, but we show in Theorem 4.6 that this axiom precisely captures the harder to define notion that a CFMM should trade “as much as possible” subject to the constraints of the other axioms.

**Axiom 6:** After executing a batch of trades, every CFMM’s spot valuations are proportional to the candidate batch valuations.

### 4.4 Consequences of Axioms

We now show that there is a unique trading rule that satisfies our axioms.

#### 4.4.1 Axioms 1 and 2 Eliminate Trading Rule A

First, observe that Axiom 1 implies that traders (who post limit orders) in batch trading systems are subject to what is commonly known as Walras’s law.

**Observation 1:** In a batch trading scheme satisfying axiom 1, a trader that initially possesses a set of assets \( x = \{ x_a \}_{a \in \mathcal{A}} \), will possess, after the batch, a set of assets \( x' \) subject to the constraint that \( p \cdot x = p \cdot x' \).

Axiom 2 implies that CFMMs must also be subject to the same law. In particular, Lemma 4.2 rules out trading rule A in Figure 1.

**Lemma 4.2:** Suppose that in a batch trading scheme satisfying Axioms 1 and 2, the market prices are \( p = \{ p_A \}_{A \in \mathcal{A}} \).

If a CFMM with initial reserves \( x \) makes a trade to \( x' \) such that \( f(x) \geq f(x') \), it must be the case that \( p \cdot x = p \cdot x' \).

This lemma shows that adding a CFMM to a batch trading scheme requires the CFMM to trade at the uniform batch exchange rates, not its own spot exchange rate (which varies with trade size). Example A.1 shows that the counterfactual, where a CFMM is forced to trade at its spot exchange rate, can render a CFMM nonfunctional in a batch.

#### 4.4.2 Axioms 4 and 5 Eliminate Trade Rule B

Theorem 4.3 shows that there is a limit to how much trading a CFMM can perform without creating an arbitrage opportunity. Specifically, it shows that a CFMM with an initial spot exchange rate without loss of generality below the batch exchange rate (between any two assets) cannot make a trade that leaves its post-batch spot exchange rate above the batch exchange rate, without leaving an arbitrage opportunity.

As a consequence, this theorem eliminates trading rule B of Figure 1.

**Theorem 4.3:** Suppose a CFMM trades between assets A and B and initially offers a spot exchange rate of \( r_0 \) from A to B. Suppose further that this CFMM trades within a batch trading scheme satisfying Axioms 1, 2, 4, 3, and 5.

Then, after execution of a batch of trades with a batch exchange rate of \( r_1 = p_A/p_B \) with, wlog, \( r_0 \leq r_1 \), the post-batch spot exchange rate \( r_2 \) of the CFMM cannot exceed \( r_1 \).

Informally, one can construct for every \( r_0 < r_1 < r_2 \) a CFMM with an initial spot rate of \( r_0 \) that cannot make a trade (at the batch rate of \( r_1 \)) to reach a spot exchange rate of \( r_2 \) while satisfying Axiom 3. A batch exchange with two CFMMs that leaves one at a spot exchange rate less than \( r_2 \) leaves an arbitrage opportunity between the two CFMMs, breaking Axiom 5.

#### 4.4.3 Categorizing the Remaining Feasible Trade Rules

We next show that the only state variables that can materially influence a feasible trading rule are a CFMM’s initial spot valuations and the batch valuations.

**Theorem 4.4:** Any trading rule satisfying Axioms 1, 2, 4, 3, and 5 can only depend on the pre-batch spot valuations of the CFMM and the batch valuations.

**Corollary 4.5:** A trading rule can be represented as a map \( F((s_A)_{A \in \mathcal{A}}, (p_A)_{A \in \mathcal{A}}) \rightarrow (s')_{A \in \mathcal{A}} \) for the initial spot valuations of the CFMM, \( p \) the batch valuations, and \( s' \) the spot valuations of the CFMM after the batch.

Informally, the argument behind Theorem 4.4 is that any state variable with material influence on the output of a trading rule must be agreed upon between any two different CFMMs; otherwise, one can build two CFMMs in one batch that differ only in this state variable, and develop a violation of Axiom 5.
Theorem 4.4, as stated, allows the trading rule to depend on the input spot valuations of a CFMM. Batch trading systems, when deployed, do not settle a single batch and then cease existence. Rather, these schemes typically settle batches of trades repeatedly, at some fixed frequency. This theorem therefore allows for the case where, prior to clearing a batch, all CFMMs agree on a shared set of spot valuations, as would be left behind from a previous batch (by Axiom 5). If, on the other hand, CFMM states are assumed to be always perturbed between batches by external events, then Theorem 4.4 would prove that a trading rule could only depend on the candidate batch valuations, and Axiom 6 would follow as a corollary of Theorem 4.4. We do not make this assumption here.

Trading rules C and D of Figure 1 satisfy Theorem 4.4, but are not the only trading rules to do so. The degenerate trading rule $F(s,p) = s$, where CFMMs make no trades at all, also satisfies the theorem.

### 4.4.4 The Unique Trading Rule (Rules C and D) Maximizes CFMM Trading

As motivation for Axiom 6, consider the following natural subclass of trading rules. Theorem 4.6 shows that the unique rule of this subclass that satisfies Axiom 6 is the rule that maximizes trading (among rules within this subclass). Lemma 4.7 shows that this rule is equivalent to Rules C and D of Figure 1.

First, assume that each CFMM cares only about the prices of assets which it trades. Assume also that the trading rule treats assets symmetrically (so e.g. a CFMM trading $A$ and $B$ behaves the same as one trading $C$ and $D$, albeit with the asset vector coordinates swapped). Finally, assume that the trading rule is invariant to redenominating the units of one asset. If one redenomimates an asset, thereby multiplying its initial spot valuation (relative to every other asset) and batch valuation by a factor of $c$, then the output spot valuation (relative to every other asset) is also multiplied by $c$.

**Theorem 4.6:** Let $F(s,p) \rightarrow s^\prime$ denote some trading rule.

Suppose that this trading rule treats assets symmetrically, and is invariant against asset redenomination (so $F(c \cdot s,c \cdot p) = c \cdot p^\prime$ for any vector $c > 0$). Suppose also that CFMMs only depend on the prices of assets which they trade.

Then $F(s,p) = s^{1-c} \cdot p^a$ for some $a \in [0,1]$.

Within this subclass, increasing $\alpha$ always implies increasing trade activity from CFMMs (by the quasi-concavity of the CFMM trading functions). $\alpha = 0$ corresponds to CFMMs that do nothing, and $\alpha = 1$ corresponds to the trading rule that sees CFMMs trade as much as possible. Thus, when $\alpha = 1$, CFMMs are doing the most they can to facilitate trading activity.

**Observation 2:** Axiom 6 (for this subclass of trading rules) is equivalent to asserting that $\alpha = 1$.

Finally, observe that the only trading rule satisfying all of our axioms is the rule $F(s,p) = p$ (i.e. $\alpha = 1$). This unique rule is equivalent to rules C and D of Figure 1.

**Lemma 4.7:** In Figure 1, trading rule C is equivalent to trading rule D.

Note that rule C is, by definition, the trading rule $F(s,p) = p$.

### 5 Integration of CFMMs in Batch Exchanges

Batch exchanges model each limit offer as an agent in an Arrow-Debreu exchange market with a particular utility function. Theorem 5.1 shows that CFMMs can also be modeled in this framework. As a consequence, batch exchanges can integrate CFMMs without changing their underlying mathematical framework, and therefore this theorem completely specifies the problem of computing equilibria in a batch exchange. This theorem is therefore crucial for the rest of our results and required for real-world deployments of batch trading schemes with CFMMs.

**Theorem 5.1:** A CFMM with trading function $f$ in a batch exchange makes the same trading decisions as a rational agent in an Arrow-Debreu exchange market using the CFMM’s reserves as its initial endowment of goods and $f$ as its utility function.

This theorem also directly incorporates trading functions that are not strictly quasi-concave or differentiable.

### 5.1 CFMM Trading Function Structure and Computational Efficiency

The asymptotic complexity of computing batch equilibria strongly depends on the design of CFMM trading functions. In the Arrow-Debreu exchange market model, natural agent utility functions can make the problem of computing an equilibrium PPAD-hard.

In a batch trading scheme, a CFMM trades at the batch rate, not by integrating over its spot exchange rate as it makes a trade (this leads to the gap between Rule A and Rule C in Figure 1). The way that an agent in an Arrow-Debreu exchange market allocates this “surplus” is a key factor determining asymptotic complexity of the equilibrium computation problem. For example, limit sell offers (Example 3.4) spend their “surplus” (the gap between batch price and limit price) by buying more, while limit buy offers (Example 3.5) respond by selling less.
Sell offers satisfy (and buy offers do not satisfy) a property known as Weak Gross Substitutability (WGS); that is to say, a market satisfies WGS if an increase in the price of one good does not cause a decrease in net demand for another good. WGS as a property can also hold or not hold with regard to the demand functions of individual agents or CFMMs, and WGS holds on a whole market if it holds on every market participant. Chen et al. [25] give a slightly stronger definition—"monotonicity"—which requires that the net demand of a good should not increase if its price increases.

It follows from Theorem 7 of Codenotti et al. [29] that equilibria can be approximated in polynomial time when every agent’s utility function satisfies WGS, but by Theorem 7 of Chen et al. [25], the problem of computing equilibria in batches where (some groups of) agents fail to satisfy WGS is PPAD-complete. For details, see Proposition B.1.

5.1.1 Efficient CFMM Trading Function Validation

Therefore, a batch exchange implementation needs an efficient way to validate that any user-supplied CFMM trading function satisfies WGS. Otherwise, adversarial users could make the equilibrium computation problem intractable and possibly shut down the exchange. The trade made by a submitted CFMM at any set of batch valuations should also be easily computable.

Our contribution in this section is a property that describes WGS in relation to the aforementioned CFMM "surplus." In an Arrow-Debreu exchange market, an agent sells its endowment \( \{e_i\} \) of goods to the market at the market valuations, to receive \( \sum e_i p_i e_i \) units of a "numeraire", and then immediately uses this "budget" to buy from the market a preferred selection of goods. The agent’s purchasing choices are "budget-invariant" if the proportion of its budget spent on any good is independent of its overall budget.

**Definition 5.2 (Budget-Invariance):** A utility function is budget-invariant if for any set of market valuations \( \{p_i\} \), and selection of goods \( \{x_i\} \) that is an optimal selection of goods at the agent’s budget \( \sum p_i x_i \), then for any constant \( \alpha > 0 \), the selection of goods \( \{\alpha x_i\} \) is optimal for a budget of \( \alpha \sum p_i x_i \).

Equivalently, for any set of CFMM reserves \( x = \{x_i\} \) and spot valuations \( p = \{p_i\} \), a CFMM’s trading function is budget-invariant if \( p \) is also a set of spot valuations at \( \alpha x \) for any \( \alpha > 0 \). Some of the theoretical literature (particularly when utilities are linear, e.g. [31]) divide one agent with several goods into several independent agents, each with one type of good. Budget-invariance classifies precisely those situations for which this division is without loss of generality.

Budget-invariance implies WGS for quasi-concave utility functions that depend on only two assets, but not for those that depend on three or more assets.

**Proposition 5.3.** Budget-invariant quasi-concave functions on two assets are always WGS.

**Example 5.4:** \( f(x, y) = x + y + xy \) is a 2-asset CFMM that is WGS, but not budget-invariant.

**Example 5.5:** \( f(x,y,z) = x^2 + 2yz \) is a budget-invariant CFMM that is not WGS.

Proposition 5.3 has two important consequences for batch exchange implementations. First, it enables a specification language that statically guarantees that a user-supplied 2-asset CFMM trading function is WGS. Specifically, a user can specify a trading function as a monotonic map from a spot exchange rate to a proportion of the CFMM’s "budget" that is spent on one good.

Second, such a specification language can guarantee efficient computation of CFMM responses to batch valuations. Iterative algorithms for computing batch equilibria in Arrow-Debreu exchange markets (often called Tâtonnement, described theoretically in [21, 29] and implemented in [56]) rely on so-called "demand queries" — for a given set of market valuations, they compute the optimal response (goods bought and sold) of every market participant to the market valuations. If the proportion of a CFMM’s budget spent on each good is efficiently computable, then the optimal response of a CFMM is efficiently computable. By contrast, if users specified a CFMM by directly giving a quasi-concave trading function, then the batch exchange implementation would need to solve a convex optimization problem for every CFMM and every demand query.

Such a specification language might be the following expressive language that contains several widely-used CFMMs.

**Example 5.6:** Let \( h(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be any polynomial with nonnegative coefficients (so \( h \) is positive and nondecreasing). A CFMM trading between \( A \) and \( B \), whose demand response is to spend a \( 1/h(p_A/p_B) \) fraction of its budget on \( B \) (and the rest on \( A \)) is budget-invariant and satisfies WGS.

1. The constant product rule \( f(a,b) = ab \) spends half of its budget on each good (\( h(\cdot) = 2 \)).
2. The weighted constant product rule \( f(a,b) = a w_a b w_b \) for positive constants \( w_a, w_b \) always spends \( \frac{w_a}{w_a + w_b} \) fraction of its budget on \( A \) (\( h(\cdot) = \frac{w_a + w_b}{w_a} \)).
Example 5.6 also points out a significant difference between CFMMs deployed in batch exchanges and those deployed in a traditional context. As $p$ increases to $\infty$, the CFMM here sells at most half of its $A$, instead of selling nearly all of it. This raises a substantial open question about the payoff of CFMM liquidity providers in batch exchanges (§8.1).

For many-asset CFMMs, batch exchanges could accept user-submitted monomial trading functions.

**Proposition 5.7.** Monomial trading functions (on any number of assets) are budget-invariant and display WGS.

### 6. A Convex Program for 2-asset WGS Utility Functions

Our contribution here is a convex program that computes equilibria in batch exchanges that incorporate CFMMs that trade between two assets and that satisfy WGS. Equivalently, this program computes equilibria in Arrow-Debreu exchange markets where every agent’s behavior satisfies WGS and every agent has utility for only two assets. The program is based off of the convex program of Devanur et al. [31] for linear exchange markets — that is, when an batch exchange contains only limit sell offers.

The key observation is that 2-asset CFMMs satisfying WGS can be viewed as (uncountable) collections of agents with linear utilities and infinitesimal endowments. This correspondence lets us replace a summation over agents with an integral over this collection of agents. However, proving correctness of our program requires a direct application of Kakutani’s fixed point theorem, instead of the argument based on Lagrange multipliers used in [31].

Smooth and strictly quasi-concave CFMMs lead to smooth objective functions. Gradients are easy to compute for many natural CFMMs. The runtime per gradient computation is linear in the number of assets and in the number of CFMMs.

#### 6.1 CFMMs as Collections of Limit Orders

We show here that a CFMM that trades two assets ($A$ and $B$) and satisfies WGS can be viewed as a pair of trading “density functions”. The batch exchange rate $p = p_A/p_B$ is always either at least or at most the CFMM’s spot exchange rate. When $p$ is greater than the spot exchange rate, the CFMM’s optimal response is to sell (in net) some amount $D(p)$ units of $A$ and purchase $pD(p)$ units of $B$. As the CFMM either in net sells $A$ or $B$ (but not both), within one batch, we divide a CFMM into two logical pieces. Each piece sells one of the assets and may purchase the other, and each piece has its own density function. In the rest of the discussion, we take the batch exchange rate $p$ to mean, for each piece, the relative value of the good sold to that of the good bought. Note that the density function depends on the CFMM’s reserves; we elide this for clarity of exposition in this section.

**Definition 6.1 (CFMM cumulative density):** Define $D(p)$ to be amount of $A$ sold to the market by a CFMM trading $A$ and $B$ in response to an exchange rate of $p$. For exchange rate levels where $A$ would be bought (i.e. for the “other half” of the CFMM), define $D(p)$ to be 0.

Define $D^{-1}(x)$ to be the least upper bound of the set $\{p : D(p) \leq x\}$. We require that $D(\infty) < \infty$ (asset amounts are finite).

This definition naturally leads to a notion of a marginal density function of a CFMM’s trades.

**Definition 6.2 (CFMM density function):** The density function $d(p)$ of a CFMM selling $A$ in exchange for $B$ is $\frac{dD(p)}{dp}$.

Observe that the density function is nonnegative if and only if the CFMM satisfies WGS (Lemma C.2). The marginal amount of $A$ sold to the market at a given exchange rate $p$ functions as a limit sell offer with limit price $p$.

A consequence of these definition is that one can view a CFMM as a collection of limit offers with some marginal density function allocating $A$ for sale along a range of exchange rates. $\int_{p_1}^{p_2} d(p)dp$ units of $A$ can be sold for $B$ with minimum exchange rates between $p_1$ and $p_2$.

Furthermore, for every nonnegative marginal density function, there always exists a CFMM with a quasi-concave trading function with that density function (Lemma C.3). As a consequence, we can add, subtract, and otherwise manipulate these density functions in natural ways as first-class primitives.

This definition assumes that $D(p)$ is well-defined, which is true only when the trading function is strictly quasi-concave (Lemma C.1). We make this assumption only for clarity of exposition. A region of $f$ where this does not hold corresponds to a jump discontinuity in $D(p)$.

Because density functions can be naturally added together, it suffices to account for a function that consists of a single jump discontinuity. For clarity of exposition, we elide this technicality in the following discussion, except to point out where a jump discontinuity would influence a theorem statement when necessary.

#### 6.2 Convex Program

When possible, we use the same notational conventions as in [31]. Let there be $N$ assets and $M$ CFMMs. The $i$th CFMM sells $A_i$ in exchange for $B_i$ and has trade density function $d_i(\cdot)$. 
Variables \( p_j \) denote the valuation of good \( j \), and the variable \( y_i \) for \( i \in [M] \) denotes the valuation-weighted trade volume of the \( i \)th CFMM and \( x_i \) corresponds to an amount of a good sold by the CFMM (so \( x_i = y_i/p_Ai \)). Define \( \beta_{i,z}(p) = \min(p_Ai,p_Bi,z) \).2 Finally, define \( g_i(x_i) \) to be \( \int_0^{D_i^{-1}(x_i)} d(p) \ln(1/p) \, dp \) \(^3\).

**Lemma 6.3:** \( g_i(\cdot) \) is a concave function, and \(-p_{Ai}g_i(y_i/p_{Ai})\) is convex.

A jump-discontinuity of size \( c \) at limit price \( p \) would have a \( g_i(x_i) = \min(c,x_i)\ln(1/p) \).

We also need a minimal set of assumptions to ensure that the price of every good at equilibrium is nonzero. Either of the following assumptions is sufficient, but neither assumption is necessary.

**Assumption 1:** For every (one-sided) CFMM density function \( D_i \), there exists \( p > 0 \) such that \( D_i(p) = 0 \).

**Assumption 2:** For every pair of CFMM density functions \( D_i,D_j \) (the two halves of a CFMM, as in Definition 6.1), \( D_i(\infty) > 0 \) and \( D_j(\infty) > 0 \).

Assumption 1 ensures that there is an exchange rate below which the CFMM is unwilling to sell a good. Assumption 2 ensures that there is always a buyer for every good (analogous to assumption * of [31]). Note that in Theorem 6.4, we assume without loss of generality that \( D_i(\infty) > 0 \) for all \( i \), but assuming Assumption 2 is not without loss of generality.

All together, we get the following convex program:

**Theorem 6.4:** The following program is convex and always feasible. Its objective value is always nonnegative. If either Assumption 1 or Assumption 2 holds, then at optimum, its objective value is 0 and optimal solutions correspond to exchange market equilibria.

\[
\begin{align*}
\text{Minimize} & \quad \sum_i p_Ai \int_0^\infty \left( \frac{d_i(z) \ln\left( \frac{p_{Ai}}{\beta_{i,z}(p)} \right)}{p_{Ai}} \right) \, dz - \sum_i p_Ai \, g_i(y_i/p_{Ai}) \\
\text{Subject to} & \quad \sum_{i : A_i = j} y_i = \sum_{i : B_i = j} y_i \quad \forall j \in [N] \\
& \quad p_j \geq 1 \quad \forall j \in [N] \\
& \quad y_i \geq 0 \quad \forall i \in [M].
\end{align*}
\]

Without loss of generality, we assume \( \int_0^\infty d_i(z) \, dz > 0 \) for all CFMMs \( i \).

**Proof.** Lemma D.1 shows the convexity and feasibility of the program. Lemma D.2 shows that the objective is nonnegative. Lemma D.3 shows that when the objective value is 0, the optimal solutions are exchange market equilibria. Finally, Lemma D.4 shows that optimal solution always has an objective value of 0. \( \square \)

A set of jump discontinuities adds a summation to the objective function, instead of an integral.

### 6.3 Rationality of Convex Program

The original program of [31] is guaranteed to have a rational solution. This program here may not, if the CFMMs can be arbitrarily constructed. However, rational solutions exist when CFMMs belong to a particular class.

**Theorem 6.5:** If, for every CFMM \( i \) in the market, the expression \( p_{Ai}D_i(p_{Ai}/p_{Bi}) \) is a linear, rational function of \( p_{Ai} \) and \( p_{Bi} \), on the range where \( D(\cdot) > 0 \), then the convex program has a rational solution.

**Example 6.6:** Some natural CFMMs satisfy the condition of Theorem 6.5.

- The density function of the constant product rule with reserves \((A_0,B_0)\) is max\(0,(pA_0-B_0)/(2p))\).
  Substitution gives (for relative exchange rates above the initial spot exchange rate) \( p_{A}\,D_i(p_{A}/p_{B}) = (p_{A}A_0-p_{B}B_0)/2 \)
- The density function of the constant sum rule \( f(a,b) = ra+b \) with reserves \((A_0,B_0) = A_0 \) if \( p > r \) and 0 otherwise.

However, this convex program cannot always have rational solutions; in fact, there exist simple examples using natural utility functions for which the program has only irrational solutions.

**Example 6.7:** There exists a batch instance containing one CFMM based on the logarithmic market scoring rule and one limit sell offer that only admits irrational equilibria.

---

\(^2\beta_{i,z}(\cdot)\) corresponds to the \( \beta \) variables of [31].

\(^3\)The expression \(-p_{Ai}g_i(y_i/p_{Ai})\) recovers an analogue of the \(-y_i\ln(u_i)\) term in [31].
7 CFMMs and Fees

We end our discussion with a remark on CFMM trading fees. Typical CFMMs deployments charge a \( \epsilon \)-percent fee on every asset transfer. Given initial reserves \( X_0, Y_0 \) and a trade \( \Delta X, \Delta Y \geq 0 \), most CFMMs (e.g. \([10, 11, 50]\)) and most CFMM research (e.g. \([12, 13]\)) require that \( X_0 Y_0 \leq (X_0 + (1-\epsilon)\Delta X)(Y_0 - \Delta Y) \).

This model naturally extends to CFMMs trading in batch exchanges. Specifically, a CFMM can charge an analogous fee on the net trade of the CFMM in the batch. In fact, accounting for such a fee can be done by modifying the CFMM trading function.

Proposition 7.1. A CFMM trading function can be modified to account for trading fees on the CFMM’s net trade in one batch.

The modification depends on the CFMM’s reserves, and so a fee-charging CFMM’s trading function would change after every batch, which could break Axiom 5.

However, note that a CFMM in a batch using this type of fee scheme only charges a fee on the net trading of the CFMM, not on every trade in the batch between e.g. \( A \) and \( B \). By contrast, when deployed as a standalone market-maker, a CFMM might receive a fee on every trade between \( A \) and \( B \) that uses the CFMM.

8 Conclusion and Open Problems

Constant Function Market Makers have become some of the most widely used exchange systems in the modern blockchain ecosystem. Batch trading has been proposed and deployed to combat some of the shortcomings of decentralized exchanges and traditional exchanges. Different implementations have taken substantially different approaches to how these two innovations should interact.

We show here from a minimal set of axioms describing the positive properties of these systems that there is a unique rule for integrating CFMMs into batch trading schemes that preserves the desirable properties of both systems together.

We also study how the structure of CFMM trading functions interacts with the asymptotics and efficient computation of market equilibria.

Finally, we construct a convex program that computes equilibria on batches of CFMMs that each trade only two assets. For many natural classes of CFMMs, the objective of this convex program is smooth and the program has rational solutions.

8.1 Open Problems

Most CFMMs deployed in practice trade between two assets, but much of the literature on CFMMs deals formally only with general, many-asset case (e.g. \([12]\)). Note that when CFMMs trade 2 assets, budget-invariance implies WGS, but not when a CFMM trades 3 or more assets. In §6, we build a convex program that integrates 2-asset CFMMs but does not obviously present a natural general, many-asset case (e.g. \([12]\)). Note that when CFMMs trade 2 assets, budget-invariance implies WGS, but not when a CFMM trades 3 or more assets. In this sense, 2-asset CFMMs appear easier to work with than 3+ asset CFMMs. Rigorously understanding the difference, if there is any, between 2-asset and 3+ asset CFMMs is an interesting line of future work.

As touched on in §7, a CFMM might receive less fee revenue when deployed in a batch trading scheme, but also trades at the batch exchange rate, not its spot exchange rate. An important question for CFMM liquidity providers is how these changes affect their expected returns on providing capital to a CFMM.

From an applied perspective, iterative algorithms like Tâtonnement \([29]\), if implemented naively, would require for each demand query an iteration over every CFMM. Ramseyer et al. \([56]\) give a preprocessing step to enable computation of the aggregate response of a group of limit sell offers in logarithmic time; identifying a subclass of CFMM trading functions that admits an analogous preprocessing step, or more generally some way of computing the aggregate demand response of a large group of CFMMs in sublinear time would be of great practical use for batch exchanges using these types of algorithms.

Typesetting on the bib items with underscores is just fucked, idk how to fix and don’t care that much

References

A Omitted Proofs from §4

A.1 Example A.1

Example A.1 shows that the strict requirement that a CFMM’s trading function be held constant (as opposed to the weaker Axiom 3) can prevent a CFMM from acting meaningfully as a market-maker, even in simple instances with few participants.

Example A.1: Consider a batch trading scheme (satisfying the axioms of §4.1) trading between two assets $A$ and $B$.

In this batch there are two limit orders and one CFMM. The first limit order sells 1 unit of $A$ for $B$ with a minimum price of 1 $B$ per $A$, and the second sells 3 units of $B$ in exchange for $A$, with a minimum price of 1/6 units of $A$ per $B$. The CFMM is a constant-product market maker ($f(a,b) = ab$) with reserves of 1 unit of $A$ and 10 units of $B$.

Then, if the CFMM follows the axioms of §4.1 and only accepts a trade from reserves $x$ to $x'$ if $f(x) = f(x')$, the CFMM cannot make any trades.

Note that the spot exchange rate of the CFMM is $10B$ per $A$, and in a classical CFMM deployment, the first limit order would strictly prefer trading with the CFMM over trading with the other limit order.

Proof. Let $r$ be the exchange rate from $A$ to $B$.

If $r < 1$, then the second limit order buys $A$ but the first does not sell $A$, nor can the CFMM sell $A$ at that price (without violating Axiom 3), so the batch trading system cannot satisfy Axiom 2.

If $r > 6$, then the first limit order sells 1 $A$ and receives $r$ units of $B$, and the second order does not sell any $B$. In this case, if the system satisfies Axiom 2, the CFMM’s reserves would be $(2A, 10-r)B$. This would violate the CFMM’s exact trading constraint, as $2*(10-r) < 2*4 = 8 < 10 = 1*10$.

In the intervening interval $(1 \leq r \leq 6)$, the first limit order sells 1 $A$ and receives $r$ $B$, while the second limit order sells 3 $B$ and receives $3/r$ $A$. The CFMM makes some trade in which it purchases $0 \leq x \leq 1$ units of $A$ and sells $rx$ units of $B$.

By the CFMM trading constraint, we must have that $10 = (1 + x)(10 - rx)$ or $10 = 10 + (10 - r)x - rx^2$. In this case, either $x = 0$ or $x = (10 - r)/r$. In the latter case, the total amount of $B$ bought by the first order must equal the amount sold by the other order plus that sold by the CFMM (by Axiom 2). Thus, $3 = r + (10 - r) = 10$, which is a contradiction.

Thus, $x = 0$, and the CFMM makes no trades.

A.2 Proof of Lemma 4.2

Lemma: Suppose that in a batch trading scheme satisfying Axioms 1 and 2, the market prices are $p = \{p_A\}_{A \in \mathcal{A}}$.

If a CFMM with initial reserves $x$ makes a trade to $x'$ such that $f(x) \geq f(x')$, it must be the case that $p \cdot x = p' \cdot x'$.

Proof. Suppose not. By Axiom 1, it must be the case that the CFMM’s reserves are subject to the constraint that $p \cdot x \geq p' \cdot x'$. This constraint in effect holds for every CFMM and every limit order (in fact, this “budget constraint” is a core component of the Arrow-Debreu exchange market model).

Since every constraint is an inequality in the same direction, and by Axiom 2, no assets can be destroyed during a batch, this constraint must be tight for every limit order and every CFMM.
A.3 Proof of Theorem 4.3

**Theorem:** Suppose a CFMM trades between assets A and B and initially offers a spot exchange rate of \( r_0 \) from A to B. Suppose further that this CFMM trades within a batch trading scheme satisfying Axioms 1, 2, 4, 3, and 5.

Then, after execution of a batch of trades with a batch exchange rate of \( r_1 = p_A/p_B \) with, wlog, \( r_0 \leq r_1 \), the post-batch spot exchange rate \( r_2 \) of the CFMM cannot exceed \( r_1 \).

**Proof.** Suppose not.

Then we can always find a CFMM trading A and B that starts with the same spot exchange rate \( r_0 \) but cannot make a sufficiently large trade (while satisfying Axiom 3) so as to reach a spot exchange rate of \( r_2 \) after the batch.

Pick some initial reserves \( (a_0,b_0) \) and some final reserves \( (a_1,b_1) \) with \( a_0r_1 + b_0 = a_1r_1 + b_1 \) for \( a_1 < a_0 \).

Let \( r_3 \) be such that \( r_2 > r_3 > r_1 \).

Define first a piecewise-linear trading curve with slope \( -r_0 \) for \( a > a_0 \), slope \( -r_3 \) for \( a_1/2 < a < a_1 \), and \( -r_2 \) on \( a < a_1/2 \) (such that the lines of slope \( -r_3 \) and \( -r_2 \) share a common endpoint). In the remaining gap, extend the lines of slope \( -r_3 \) and \( -r_0 \) to meet.

If one insists on a strictly differentiable trading curve, then one can approximate this piecewise-linear function arbitrarily closely by a smooth curve while ensuring that the curve still passes through \( (a_0,b_0) \) and \( (a_1,b_1) \) with the same derivatives at those points.

We can then extend this trading curve to a total trading function \( f \) by rescaling the trading curve. For any point \( x = (a,b) \), define \( x' \) to be the point on the original trading curve at which the ray from the origin through \( x \) intersects the curve. Then we can define \( f(x) = |x|/|x'| \).

Given the above axioms, a CFMM with this trading function cannot make sufficient trades to move to a point at which its spot exchange rate exceeds \( r_1 < r_2 \), as it can trade at most to reach the point \( (a_1,b_1) \) with spot exchange rate \( r_3 < r_2 \).

□

A.4 Proof of Theorem 4.4

**Theorem (Theorem 4.4):** Any trading rule satisfying Axioms 1, 2, 4, 3, and 5 can only depend on the pre-batch spot valuations of the CFMM and the batch valuations.

Informally, the argument below shows that the trading rule cannot depend on any information that differs between two CFMMs. The information allowed by the theorem statement is the information that a batch exchange either axiomatically guarantees to be the same between any application of the trading rule (the batch valuations) or what an implementation might guarantee to be the same (the pre-batch spot valuations). Note that we do not assume, in the end, that CFMMs must agree upon a set of spot valuations prior to execution of a batch of trades, but leave the possibility open here for completeness of exposition.

**Proof.** Consider a CFMM trading between a set of assets \( A \in \mathcal{A} \). By Axiom 4, any trading rule can be described as a function \( F \) of the CFMM’s reserves, its trading function, and the batch valuations.

Suppose that the theorem does not hold. Then there exist two trading functions \( f_1, f_2 \), a set of batch valuations \( \{p_A\}_{A \in \mathcal{A}} \) and two sets of reserves \( \hat{x}_1, \hat{x}_2 \) such that the spot valuations of a CFMM with trading function \( f_1 \) and reserves \( \hat{x}_1 \) are proportional to the spot valuations of a CFMM with trading function \( f_2 \) and reserves \( \hat{x}_2 \) (recall that the valuations may differ by a constant factor, as the exchange rates are quotients of valuations), but where \( F(f_1, \hat{x}_1, q) \) is not proportional to \( F(f_1, \hat{x}_1, q) \).

Then consider a batch exchange instance in which there are only two CFMMs, one with trading function \( f_1 \) and reserves \( \hat{x}_1 \) and the other with trading function \( f_2 \) and reserves \( \hat{x}_2 \). These two CFMMs agree on an initial set of spot valuations. However, in this batch exchange instance, the trading rule \( F \) would output different sets of spot valuations for each CFMM, thereby violating Axiom 5.

Thus, \( F \) may depend only on the set of batch valuations and the initial set of spot valuations.

□

A.5 Proof of Theorem 4.6

**Theorem:** Let \( F(s,p) \rightarrow s' \) denote some trading rule.

Suppose that this trading rule treats every asset identically, and is invariant against asset redenomination (so \( F(c \cdot s.c \cdot p) = c \cdot p' \) for any vector \( c > 0 \)).

Then \( F(s,p) = s^{1-\alpha}p^\alpha \) for some \( \alpha \in [0,1] \).

**Proof.** Because the trading rule treats every asset identically and CFMMs only depend on the assets in which they trade, to specify the interaction between any set of assets, it suffices to specify the interaction between any two assets.
Batch Exchanges with Constant Function Market Makers:
Axioms, Equilibria, and Computation

Observe first that by assumption, for any constant \( \lambda \in \mathbb{R}_{>0} \), if \( F(s,p) = s' \), then \( F(\lambda s, \lambda p) = \lambda s' \). As such, when considering only two assets, only the relative valuations of the assets matter. Therefore, one can specify \( F \) by a function \( g(s_A/s_B, p_A/p_B) = s_A'/s_B' \).

Axiom 5 requires that \( g(s_A/s_B, p_A/p_B) + g(s_B/s_A, p_B/p_A) = g(s_A, p_A) + g(s_B, p_B) \).

To simplify the picture, consider the case where \( s_A = 1 \) for all assets \( A \in \mathcal{A} \). Then we can define the function \( \hat{g}(t) = g(1,t) \). Note that \( \hat{g}(1) \) must be 1.

This function is subject to the condition that \( \hat{g}(\Pi t_i) = \Pi_i \hat{g}(t_i) \) for any sequence of values \( t_i \).

Finally, define \( h(t) = \hat{g}(e^t) \). Thus \( h(\Sigma t_i) = \Pi_i h(t_i) \).

Observe that \( \frac{\partial}{\partial t} h(t_1 + t_2) = h'(t_1 + t_2) = h(t_2) h'(t_1) \), and similarly that \( \frac{\partial^2}{\partial t^2} h(t_1 + t_2) = h'(t_1 + t_2) = h(t_1) h'(t_2) \).

Taking \( t_1 = 0 \) gives \( h'(t_2) = h(t_2) h'(0) \).

Repeated differentiation gives \( h^n(t) = h'(0)^n h(t) \).

Writing the Taylor expansion of \( h \) gives

\[
h(t) = \sum_{n=0}^{\infty} \frac{(t h'(0))^n h(0)}{n!} = e^{t h'(0)}
\]

Substituting back in \( x = e^t \) gives

\[
\hat{g}(x) = x^{\sqrt{\alpha}}(1)
\]

By Theorem 4.3, we must have that \( \hat{g}(x) = x^\alpha \) with \( \alpha \in [0,1] \).

Thus, \( \hat{g}(1,t) = t^\alpha \) for some \( \alpha \).

Note that

\[
g(a,b) = g(1+a, b/a + a) = a \cdot g(1, b/a) = a \cdot (b/a)^\alpha = a^{1-\alpha} b^\alpha
\]

completing the proof.

A.6 Proof of Lemma 4.7

**Lemma:** In Figure 1, trading rule C is equivalent to trading rule D.

**Proof.** The trading function \( f \) is quasi-convex. Thus, to maximize \( f \) on a hyperplane (as in rule D) is to find the point where the gradient of \( f \) is normal to the hyperplane. But the gradient of \( f \) at a point is equal to the spot exchange rates at said point (up to rescaling), so when \( f \) is tangent to the hyperplane defined by the batch market prices, the spot exchange rates of the CFMM equal the batch market prices (as in rule C).


B Proofs of §5

**Theorem** (Theorem 5.1): A CFMM with trading function \( f \) in a batch exchange makes the same trading decisions as a rational agent in an Arrow-Debreu exchange market using the CFMM’s reserves as its initial endowment of goods and \( f \) as its utility function.

**Proof.** A CFMM satisfying the requisite axioms makes an exchange with the batch at the market prices until its spot price equals the market prices.

At the point \( x \) where the spot prices of the CFMM equal the market prices, the curve \( f(x) = f(x) \) is by construction tangent to the hyperplane containing \( x \) (and the CFMM’s initial reserve point) normal to the market price vector. By the quasi-convexity of \( f \), this point maximizes \( f \) along the this hyperplane. As such, a rational agent maximizing \( f \) given the CFMM’s initial reserves will also choose to trade until its reserves are at \( x \).

**Proposition** (Proposition 5.3): Budget-invariant CFMMs trading between two assets are always WGS.

**Proof.** Let \( f(x,y) \) be the trading function of a budget-invariant CFMM.

Suppose that at the current market prices of \( p_X, p_Y \), the point \( (x_0, y_0) \) is an optimal bundle of goods.

Now suppose that \( p_X \) increases to \( p_X' > p_X \).

By the quasiconvexity of \( f \), every point \( (x,y) \) that minimizes \( p'_X x + p_Y y \), subject to the constraint that \( f(x,y) \geq f(x_0, y_0) \), must have \( x \leq x_0 \). As such, \( x/y \leq x_0/y_0 \).
Again by the quasiconvexity of \( f \) a point \((x,y)\) maximizes \( f \) subject to some budget constraint (one that is less than the budget constraint of \( p'Xx_0 + py_0 \)).

Because \( f \) is budget-invariant, a point \((x_1, y_1)\) that maximizes \( f \) subject to the actual budget constraint \( p'_Xx_0 + py_0 \) must be a scalar multiple of a point that maximizes \( f \) subject to a lower budget constraint. This means that \( x_1/y_1 \leq x_0/y_0 \).

In other words, an increase in the price of \( X \) causes the budget to not decrease and causes the amount of \( Y \) purchased relative to \( X \) to not decrease, so the overall demand for \( Y \) from the CFMM cannot decrease.

Hence, the CFMM satisfies WGS. □

**Example** (Example 5.4): \( f(x,y) = x + y + xy \) is a 2-asset CFMM that is WGS, but not budget-invariant.

**Proof.** The spot price of this CFMM, in \( X \) relative to \( Y \), is \( 1 + \frac{y}{1+x} \), which is not constant given a uniform multiplicative increase in \( x \) and \( y \). As such, this function cannot be budget invariant.

However, an optimal bundle \((x_0, y_0)\) in response to prices \( p_X \) and \( p_Y \) must have \( p'_X/x_0 = 1 + \frac{y_0}{1+x_0} \).

Now suppose that the price of \( X \) increases to \( p'_X \).

By Walras’s Law, an optimal response \((x_1, y_1)\) to \( p'_X, p_Y \) cannot have \( x_1 < x_0 \) and \( y_1 < y_0 \).

Because \( p'_X/x_1 = 1 + \frac{y_0}{1+x_0} \), we must have that \( y_1 \geq y_0 \), so the net demand for \( Y \) does not decrease. □

**Example** (Example 5.5): \( f(x,y,z) = x^2 + yz \) is a budget-invariant CFMM that is not WGS.

**Proof.** Changing every asset amount by a constant factor does not change the direction of the gradient of \( f \) (although it may rescale the magnitude). As such, for any feasible collection of goods \((x,y,z)\), an increase in the budget constraint by a factor of \( a \), given the same relative prices of \( X, Y, \) and \( Z \), enables the CFMM to purchase a corresponding uniformly increased collection of goods \((ax,ay,az)\). \((x,y,z)\) is optimal at the original budget if and only if \((ax,ay,az)\) is optimal at the increased budget. Hence, the CFMM is budget-invariant.

For sufficiently large \( p_Y \), this CFMM spends no money on \( Y \) or \( Z \), but does purchase both for sufficiently small \( p_Z \) and \( p_Y \). Hence, an increase in the price of \( Y \) can cause a decrease in the demand for \( Z \), so the CFMM does not satisfy WGS. □

**Example** (Example 5.6): Let \( h(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) be any polynomial with nonnegative coefficients (so \( h \) is positive and nondecreasing). A CFMM trading between \( A \) and \( B \), whose demand response is to spend a \( 1/h(p_A/p_B) \) fraction of its budget on \( B \) (and the rest on \( A \)) is budget-invariant and satisfies WGS.

1. The constant product rule \( f(a,b) = ab \) spends half of its budget on each good \((h(\cdot) = 2)\).
2. The weighted constant product rule \( f(a,b) = a^{w_a}b^{w_b} \) for positive constants \( w_a, w_b \) always spends a \( \frac{w_a}{w_a + w_b} \) fraction of its budget on \( A \) \((h(\cdot) = \frac{w_a}{w_a + w_b})\).

**Proof.** The spot exchange rate from \( A \) to \( B \) of the weighted constant product rule \( f(a,b) = a^{w_a}b^{w_b} \) is \( p = \frac{w_b}{w_a} \). Therefore, at any batch valuations \((p_A, p_B)\), it must be the case that \( p_A w_A a = p_B w_B b \). The CFMM’s budget at these valuations is \( p_A a + p_B b = \tilde{K} \). Thus, \( p_A a + p_B w_A a = \tilde{K} \), so \( p_A a = \tilde{K}(1 + \frac{w_A}{w_B}) = \tilde{K} \frac{w_B}{w_A + w_B} \).

The case of the constant product rule follows from setting \( w_a = w_b = 1 \). □

**Proposition** (Proposition 5.7): Monomial trading functions (on any number of assets) are budget-invariant and display WGS.

**Proof.** Suppose that some trading function has the form \( f(x) = \Pi x_i^{d_i} \). Then, at any price point \( p \), the CFMM’s optimal response has it move to a point where \( \frac{p_i}{p} = x_i^{d_i} \) for any two assets \( i,j \), or equivalently, \( p_{ix}d_i = p_{ij}d_j \).

This function is budget-invariant because the relative purchasing of any asset to any other depends only on the ratio of the asset prices. That is to say, a uniform multiplicative increase in every asset amount does not change the direction of the gradient of the function.

An increase in the price of one good \( i \) may increase the overall budget of the agent. By the above equation, it also must increase the fraction of the budget the agent spends on every good other than \( i \). As such, an increase in \( p_i \) cannot cause a decrease in demand for any other good, so the monomial must satisfy WGS. □
Proposition B.1. The problem of computing equilibria in batches of sell offers is solvable in polynomial time, but the problem of computing equilibria in batches that include buy and sell offers is PPAD-complete.

More generally, exchange market equilibria can be approximated in polynomial time if the behavior of every market participant satisfies WGS. However, the problem of computing equilibria in markets that can contain even one type of non-WGS utility function is PPAD-complete.

Proof. Limit sell offers generate linear utility functions on two assets — functions of the form \( f(x,y) = \alpha_x x + \alpha_y y \). Limit buy offers generate linear utility functions on two assets with threshold — functions of the form \( f(x,y) = \alpha_x x + \min(\alpha_y y, c) \).

Linear utility functions satisfy WGS. Approximate computability in polynomial time follows from Theorem 7 of [29].

The hardness result is Corollary 2.1 of [25], which follows from Theorem 7 of [25]. The theorem follows by constructing two types of sub-markets—“normalized non-monotone markets” and “price-regulating markets”—and linking them with “single-minded” traders.

Example 2.4 of [25] shows how to construct non-monotone markets using only the above types of utility functions, where each non-monotone market contains only two goods. The price-regulating markets have only linear utility functions. The construction builds one price-regulating market for each non-monotone sub-market, trading the same set (and thus same number, 2) of goods — so again, one can build these markets from limit sell offers (it also builds an extra price-regulating market that also trades only two goods).

The “single-minded” traders sell all of their goods for another good at any price—as though their limit prices were 0. These traders sell either one or two types of goods at once, so they do not exactly correspond to limit sell offers, but it is without loss of generality to replace an agent with a linear utility function that sells two types of goods with two separate agents, each of which uses the same utility function but where each only sells one of the original goods.

\[ \square \]

C Proofs of §6.1

C.1 Supporting Lemmas for §6.1

For completeness, we give here some basic lemmas supporting the discussion in §6.1.

Lemma C.1: A CFMM density function \( D(p) \) is well-defined if the trading function \( f \) is strictly quasi-concave.

Proof. By Theorem 5.1, the optimal response of a CFMM with reserves \((a_0, b_0)\) and trading function \( f \) to an exchange rate \( p \) is to trade to a set of reserves \((a,b)\) that maximizes \( f \) subject to the constraint that \( pa_0 + b_0 = pa + b \). As such, there exists \( K \) such that \((a,b)\) minimizes \( f \) subject to the constraint that \( f() = K \). As \( f \) is strictly quasi-concave, such a point is unique.

Lemma C.2: The marginal density function \( d(p) \) is nonnegative if and only if the CFMM satisfies WGS.

Proof. If there is some \( p \) such that \( D(p) \) is decreasing at \( p = p_A/p_B \), then there is some \( p' \) such that \( p'_B > p_B \) where \( D(p) < D(p_A/p'_B) \).

In other words, an increase in the value of \( B \) caused a decrease in the CFMM’s demand for \( A \) (the CFMM keeps less \( A \)), so the CFMM does not satisfy WGS.

If \( D(p) \) is always nondecreasing, then for every \( p = p_A/p_B \) and every \( p'_B > p_B \), \( D(p_A/p'_B) \leq D(p) \), so the CFMM’s demand for \( A \) is nondecreasing as the market valuation of \( B \) increases. Observe that from the perspective of the CFMM (which looks only at the exchange rate from \( A \) to \( B \)), not the valuations \( p_A \) or \( p_B \), any change in the spot valuation of \( A \) can be equivalently described by a change in the spot valuation of \( B \).

Lemma C.3: Consider a pair of nondecreasing density functions \( D_1() \) and \( D_2() \), where \( D_1 \) sells \( A \) for \( B \) and \( D_2 \) sells \( B \) for \( A \) such that \( D_1^{-1}(0) \geq 1/D_2^{-1}(0) \). Then there must exist a quasi-concave utility function \( f \) that a CFMM using this function and trading \( A \) and \( B \) induces density \( D_1 \) when selling \( A \) for \( B \) and \( D_2 \) when selling \( B \) for \( A \).

Proof. Let the initial reserves of the CFMM be \( a_0 = D_1(\infty) \) units of \( A \) and \( b_0 = D_2(\infty) \) units of \( B \). Let \( p_0 = D_1^{-1}(0) \). Consider the set of points \((a,b)\) such that \( a(p) = a_0 - D_1(p) \), \( b(p) = b_0 + pD_1(p) \) for all \( p \geq p_0 \), and \( a(p) = a_0 + (1/p)D_2(1/p) \), \( b(p) = b_0 - D_2(1/p) \) for all \( p \leq p_0 \). These points are the possible reserves of the CFMM after making an optimal trade, in response to any batch exchange rate \( p \) from \( A \) to \( B \).

It suffices to show that there is a quasi-concave function \( f \) for which these points are optimal trading responses. That is, for any \( p, f \) is maximized along the line \( pa + b = pa_0 + b_0 \) at \((a(p), b(p))\). Observe that because \( D_1 \) and \( D_2 \) are continuous, this set of points defines a curve in the nonnegative orthant. Furthermore, because \( D_1 \) and \( D_2 \) are nondecreasing, then \( a(p) \) is nonincreasing and \( b(p) \)

\[ \text{The approximation is to a (1+\varepsilon) level of accuracy for any \( \varepsilon > 0 \) and the natural metric of accuracy given in Definition 1 of [29].} \]
is strictly increasing as \( p \) increases for \( p > p_0 \), and \( b(p) \) is nonincreasing and \( a(p) \) is strictly increasing as \( p \) decreases for \( p < p_0 \). As such, this curve divides the nonnegative orthant into two pieces (above and below the curve). We define \( f \) separately on each piece.

First, define \( l(p) \) to be the ray defined by the constraints \( pa + b = pa_0 + b_0 \) and \( a < a_0 \) for \( p > p_0 \) or \( b > b_0 \) for \( p < p_0 \). Observe that \((a(p), b(p)) = (a_0, b_0)\) for all \( p_0 > p > 1/p_0 \). Observe also that the rays \( l(p), l(1/p) \) demarcates a cone, and that the cone defined by \( p_1 \) strictly contains that defined by \( p_2 \) when \( p_2 > p_1 > p_0 \).

Define, at any point \((a(p), b(p))\), the function \( f \) to be \( p \) for \( p \geq p_0 \) and \( f \) to be \( 1/p \) for \( p < 1/p_0 \). Set \( f(a_0, b_0) = p_0 \). Clearly, \( f \) is well-defined for every point \((a(p), b(p))\).

Then, for every \( p > p_0 \), define \( f \) to be \( p \) on the line segment interpolating between \((a(p), b(p))\) and \((a(1/p), b(1/p))\). These line segments cannot intersect. For any \( p_1, p_2 \) with \( wog \) \( p_2 > p_1 > p_0 \), the line segment corresponding to \( p_1 \) runs between \( l(p_1) \) and \( l(1/p_1) \), strictly between which run \( l(p_2) \) and \( l(1/p_2) \). It suffices, therefore, that \( b(p_2) > b(p_1) \) and \( a(p_2) > a(p_1) \).

Then, for each \( p > p_0 \), define \( f \) to be \( p \) for every \((a,b)\) on \( l(p) \) with \( a < a(p) \) and for every \((a,b)\) on \( l(1/p) \) with \( b < b(1/p) \). Finally, for every point left as yet undefined, define \( f \) to be 0. Observe that the level sets of \( f \) are convex, so \( f \) is concave.

Observe that \((a(p), b(p))\) maximizes \( f \) on the line defined by \( pa + b = pa_0 + b_0 \), as \( p \) is a spot exchange rate at \((a(p), b(p))\).  

A single jump discontinuity (i.e. \( D(p) = 1 \) if \( p > p_0 \) and 0 otherwise) would correspond to a linear utility function. Such discontinuities would make \( D_1(\cdot), D_2(\cdot), a(\cdot), \) and \( b(\cdot) \) into set-valued functions in the above proof, but would not change the construction or the argument.

### C.2 Proofs of Examples

**Example** (Example 6.6): Some natural CFMMs satisfy the condition of Theorem 6.5.

- **Constant Product Rule**
  
The density function of a CFMM using the constant product rule with reserves \((A_0, B_0)\) is \(\max(0,(pA_0 - B_0))/(2p))\).

Substitution gives (for relative exchange rates above the initial spot exchange rate) \(pA_1 D_l(pA_1/pB_1) = (pA_1 A_0 - pB_1 B_0)/2\)

- **Constant Sum Rule**
  
The density function of a CFMM using the constant sum rule \(f(a,b) = ra + b\) with reserves \((A_0, B_0)\) is \(A_0\) if \( p > r \) and 0 otherwise.

**Proof.**

1. Given a batch exchange rate of \( p \) units of \( B \) per \( A \), a CFMM using the constant product rule must make a trade so that its reserves \((A_1, B_1)\) after trading satisfy the following two conditions. Without loss of generality, assume \( p \) is greater than the CFMM’s current spot exchange rate of \( B_0/A_0 \), so the CFMM sells \( A \) to the market and purchases \( B \).

First, the spot exchange rate of the CFMM, \( B_1/A_1 \), should be equal to \( p \). And second, the CFMM must trade at the batch exchange rate, so \((A_0 - A_1)p = (B_1 - B_0)\). Thus \( D(p) = A_0 - A_1 \).

Combining these two, we get

\[
\frac{B_0 + pd(p)}{A_0 - D(p)} = p
\]

Solving for \( D(p) \) gives

\[
D(p) = \frac{pA_0 - B_0}{2p}
\]

2. The CFMM sells its entire endowment if and only if the offered exchange rate is at least the constant exchange rate \( r \).

**D Proofs of §6.2**

**Lemma** (Lemma 6.3): \( g_j(x) \) is a concave function (and \(-p_{A_j} g_j(y_i/p_{A_j})\) is convex).

**Proof.** Observe that
Axioms, Equilibria, and Computation

19

as

bound the equilibrium with nonzero prices.

of convex terms, corresponding to jump discontinuities).

Lemma D.1: The so-called convex program of §6.2 is convex and feasible.

Proof. Each term \( d_i(z) \ln(p_{A_i}/\beta_{i,z}(p)) \) is convex, and the integral of convex functions is convex. (the same goes for a summation of convex terms, corresponding to jump discontinuities).

Feasibility follows from setting \( y_i = 0 \) for all \( i \).

As an observation, note that [31] requires a technical condition (condition ∗) to ensure feasibility and that there exists a market equilibrium with nonzero prices.

If density functions are viewed as an infinite collection of limit sell offers, then condition ∗ is satisfied if either Assumption 1 or 2 holds.

Informally, we avoid infeasibility here by combining the first and second constraints of the program of [31], and effectively upper bound the \( y_i \) variables through the utility calculation in the \( g_i \) function.

In the remainder of the proofs of this section, isolated jump discontinuities turn density function into set-valued functions (so e.g. statements of the form \( y_i = p_{A_i}D_i(p_{A_i}/p_{B_i}) \) become \( y_i \in p_{A_i}D_i(p_{A_i}/p_{B_i}) \), and integrals over a range of exchange rates must account for the discontinuity with an extra term. For example, if \( D(p) \) has a discontinuity of size \( c \) at exchange rate \( r \), then an integral such as \( \int_0^{D^{-1}(x)} f(p)dp \) becomes, for e.g. \( D^{-1}(x) = r. \int_0^r f(p)dp + f(r)(x - \lim_{p \to r-}(D(p))) \).

Lemma D.2: The objective value of the convex program is nonnegative.

Proof. By construction, for all \( z, \ln(\beta_{i,z}(p)) \leq \ln(z) + \ln(p_{B_i}) \), or alternatively, \( \ln(1/z) \leq -\ln(\beta_{i,z}(p)) + \ln(p_{B_i}) \).

Rearranging the second half of the objective gives

\[
\sum_i P_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) \ln(1/z) dz 
\leq \sum_i P_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) (\ln(\beta_{i,z}(p)) + \ln(p_{B_i})) dz 
\]

Note that \( P_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) dz = P_{A_i} (y_i/p_{A_i}) = y_i \).

Applying the first program constraint gives

\[
\sum_i \left( P_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) \ln(p_{B_i}) dz \right) 
= \sum_{A \in [N]} \sum_{B_i \in A} y_i \ln(p_{A}) 
= \sum_{A \in [N]} \sum_{\lambda \in A} y_i \ln(p_{A}) 
\]

Hence,

\[
\sum_i P_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) \ln(1/z) dz 
\leq \sum_i P_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) (\ln(p_{A_i}) + \ln(\beta_{i,z}(p))) dz 
\]
As $\beta_{i,z}(p)$ is at most $p_{A_i}$, the term inside the integral is nonnegative for any $z$, and thus this summation is upper bounded by

$$\sum_i p_{A_i} \int_0^\infty d_i(z) \left( \ln \left( \frac{p_{A_i}}{\beta_{i,z}(p)} \right) \right) dz$$

which is the first term of the objective.

\[ \square \]

**Lemma D.3**: The objective of the convex program is 0 if and only if $y_i = p_{A_i} D_i(p_{A_i}/p_{B_i})$ for all $i$.

**Proof.** View the objective function as a first (nonnegative) term from which a second (nonnegative) term is subtracted. Define $z_{0,i} = p_{A_i}/p_{B_i}$.

Following the rewrites of the previous proof, the overall objective is lower bounded by

$$\sum_i p_{A_i} \int_0^{z_{0,i}} d_i(z) \ln \left( \frac{p_{A_i}}{\beta_{i,z}(p)} \right) dz - \sum_i p_{A_i} \int_0^{D_i^{-1}(y_i/p_{A_i})} d_i(z) \ln \left( \frac{p_{A_i}}{\beta_{i,z}(p)} \right) dz$$

where the integrals in the first summation run to $z_{0,i}$ instead of $\infty$ because the body of the integral is 0 for $z \geq z_{0,i}$.

Note $\ln(p_{A_i}/\beta_{i,z}(p))$ is always nonnegative and strictly positive for $z < p_{A_i}/p_{B_i}$, and that the inequality $\ln(\beta_{i,z}(p)) \leq \ln(z) + \ln(p_{B_i})$ is an equality if and only if $z = z_{0,i} = p_{A_i}/p_{B_i}$.

If $y_i = p_{A_i} D_i(p_{A_i}/p_{B_i})$ for all $i$, then this lower bound on the objective is tight, and thus the objective value is 0.

If there are any $i$ such that $y_i < p_{A_i} D_i(p_{A_i}/p_{B_i})$, then the upper bound for the second term of the objective is strictly smaller than the first term. If there are any $i$ such that $y_i > p_{A_i} D_i(p_{A_i}/p_{B_i})$, then the second term of the objective is strictly smaller than its upper bound. If either case holds, then the objective cannot be 0.

What remains to prove is the following:

**Lemma D.4**: The optimal value of the convex program is 0, if either Assumption 1 or Assumption 2 holds.

This part of the argument differs from that of [31]. The KKT conditions for this program are substantially more complicated and more difficult to work with, so we take an alternative approach.

It suffices to show that there exists an equilibrium of an exchange market (where again, each agent trades between only two goods and every agent’s demand response satisfies WGS).

First, consider the graph $G$ where each vertex corresponds to one asset and each edge corresponds to one CFMM trading the assets at its endpoints. We assume without loss of generality that this graph is connected. If not, then we can apply the following argument to each connected subgraph, and then combine the resulting equilibria together to get one unified equilibrium where all assets have positive prices.

Let $P_n$ denote the price simplex on $n$ assets.

We define first the Aggregate Demand of a set of CFMMs to be a function $Z : P_n \to \mathbb{R}^n$ that denotes the total amount of each asset sold by CFMMs to the market minus the amount bought by CFMMs from the market, when each CFMM makes its optimal trade in response to prices $p$. Market equilibria are exactly the points $p$ such that $p_A Z_A(p) = 0$ for all assets $A$ and $Z_A(p) \leq 0$ - that is to say, if an asset’s price is nonzero, then its net demand is 0.

Defining an aggregate demand function takes some care in a general Arrow-Debreu market. In order for $Z(p)$ to be meaningful, it must encode the optimal responses of each agent to market prices. But the optimal response of an agent to market prices may not always be continuous or even well defined. Our aggregate demand function has to be a set-valued function, to account for the case where a CFMM is indifferent between multiple options.

**Definition D.5**: Define the demand function of a CFMM $i$ to be $Z_{i,A_i}(p) = -D_i(p_{A_i}/p_{B_i})$, $Z_{i,B_i}(p) = -(p_{A_i}/p_{B_i}) Z_{i,A_i}(p)$, and 0 for other assets.

The aggregate demand function of a set of CFMMs is the sum of the demand functions of each CFMM.

Unfortunately, $Z_i(p)$ is still not well-defined when $p_{B_i} = 0$.

Instead, we can define $\tilde{Z}_i(p)$ to be a lightly modified defined version of the aggregate demand function.

First, define $E_A$ to be an amount greater than the total amount of asset $A$ present in the system. That is, $E_A = 1 + \sum_{A_i \neq A} D_i(\infty)$. Note that the asset amounts traded within a batch system are finite, so $D_i(\infty) < \infty$. Define $E = \max_A \{E_A\}$. 


**Definition D.6:** For \( p_A \) and \( p_B \) nonzero, define 
\[
\alpha_i(p) = \max(-D_i(p), -(p_B / p_A)_E) \quad \text{and} \quad \beta_i(p) = -(p_A / p_B)_E \alpha_i(p) = \min((p_A / p_B) D_i(p), E).
\]
Define \( \hat{Z}_{i,A_i}(p) = \alpha_i(p) \) and \( \hat{Z}_{i,B_i}(p) = \beta_i(p) \).

Observe that as \( p_B \) goes to 0, \( \beta_i(p) \) approaches \( E \), and that as \( p_A \), approaches 0, \( \alpha_i(p) \) approaches \( -D_i(\infty) \) (by assumption, \( D_i(\infty) > 0 \) for all \( i \)).

When \( p_B = 0 \) and \( p_A > 0 \), define \( \hat{Z}_{i,B_i} = E \) and \( \hat{Z}_{i,A_i} = 0 \).

When \( p_A = 0 \) and \( p_B > 0 \), define \( \hat{Z}_{i,B_i} = 0 \) and \( \hat{Z}_{i,A_i} = -D_i(\infty) \).

When \( p_A = p_B = 0 \), define \( \hat{Z}_{i,A_i} = \{ x : -D_i(\infty) \leq x \leq 0 \} \) and \( \hat{Z}_{i,B_i} = \{ x : 0 \leq x \leq E \} \).

Define \( \hat{Z}(p) \) to be \( \Sigma_i \hat{Z}_i(p) \).

Note that by construction, for all \( z \in \hat{Z}(p), z \cdot p = 0 \).

**Proof.** Consider the following two-agent game. The state space of the game is a tuple \((z, p)\), where the first player chooses any strategy \( z \in \hat{Z}(p) \) and the second player chooses strategy \( p \in P_n \).

The second player receives payoff \( z \cdot p \).

Note that the components of \( \hat{Z}(p) \) are all bounded; as such, there exists a hypercube \( X \) such that \( \hat{Z}(p) \subset X \) for all \( p \in P \).

These payoffs induce a mapping from states to payoff-maximizing best responses. The graph of this mapping is closed (because \( \hat{Z}(p) \) has a closed graph) and the image of each point under this mapping is nonempty and convex (as \( \hat{Z}(p) \) is always convex and nonempty).

By Kakutani’s fixed point theorem [46], this mapping has a fixed point. In other words, there exists a point \( p \in P \) and a point \( z \in \hat{Z}(p) \) such that for all goods \( A, z_A \leq 0 \) and if \( z_A < 0 \), then \( p_A = 0 \).

(Suppose there is a set of goods \( G \) with \( z_g > 0 \) for all \( g \in G \). Then every optimal response for the second player sets the prices of some of these goods to positive values, and all other prices to 0. Let the set of goods with positive prices be \( H \). Note that every CFMM preserves Walras’s law, i.e. \( p \cdot \hat{Z}(p) = 0 \). So the CFMMs buying goods inside \( H \) selling goods outside \( H \) cannot make any purchases, and those trading goods within \( H \) satisfy Walras’s law. Thus, \( \Sigma_i \hat{Z}_{i,A_i} \cap H \hat{Z}(p) = 0 \). Thus, some good \( h \in H \) must have \( z_h < 0 \), which is a contradiction).

Let \( H \) be the set of goods \( h \) such that \( p_h = 0 \).

- If Assumption 2 holds, because the trading graph \( G \) is connected and because \( p \in P \) (so there always exists at least one good with a positive price), there must be some good \( A \) with \( p_A > 0 \) and a CFMM that sells \( A \) in exchange for some good \( B \in H \).

But then \( \hat{Z}_{i,B} = E \). As such, we must have that for all \( z \in \hat{Z}(p), z_B > 0 \), which is a contradiction.

- If Assumption 1 holds, then for every good \( h \in H \), there exists a positive price \( p'_h \) such that every CFMM selling \( h \) still sells 0 units of \( h \). Either there exists some CFMM that sells a good with positive price for a good in \( H \) (so the argument as in the case of Assumption 2 holds) or one can build a new equilibrium by replacing \( p_h \) with \( p'_h \).

Hence, there exists an equilibrium where every price is nonzero.

Finally, it suffices to show that for each CFMM \( i \), at equilibrium, \( \hat{Z}_{i,A_i} = Z_{i,A_i}(p) \). This is true precisely when \( \beta_i(p) < E \), which must hold (or else \( \hat{Z}_{i,B} > 0 \) by construction).

Therefore, there exists a market equilibrium with exclusively nonzero prices.

\[ \Box \]

**D.1 Proof of Theorem 6.5**

**Theorem:** If for every CFMM \( i \) in the market, the expression \( p_A, D_i(p_A, p_B) \) is a linear, rational function of \( p_A \) and \( p_B \), on the range where \( D(\cdot) > 0 \), then the convex program has a rational solution.

**Proof.** At an optimal point \((p, y)\), for every CFMM \( i \), it must be the case that \( p_A, D_i(p_A, p_B) = y_i \) (or else the \((p, y)\) would not be a market equilibrium).

If \( D_i \) satisfies the requirements of the theorem, then there is some linear function \( q_i \) for which \( y_i = q_i(p_A, p_B) \) when \( y_i \geq 0 \).

To the set of existing constraints in the convex program, add for each \( i \) with \( y_i > 0 \) the constraint that \( y_i = q_i(p_A, p_B) \). For each \( i \) with \( y_i = 0 \), add the constraints that \( y_i = 0 \) and that \( q_i(p_A, p_B) \leq \lim_{r \to y_i} r D(r) \).

This system of constraints is clearly satisfiable, and every point in the system is in fact a market equilibrium (every point satisfies \( p_A, D_i(p_A, p_B) = y_i \)).

Each of the constraints is linear and rational, so these constraints define a rational polytope. The extremal points of this polytope must therefore be rational.

\[ \Box \]
D.2 Proof of Example 6.7

**Example:** There exists a batch instance containing one CFMM based on the logarithmic market scoring rule and one limit sell offer that only admits irrational equilibria.

**Proof.** Consider a batch instance trading two assets $A$ and $B$ that contains one CFMM and one limit sell offer. The CFMM uses the trading rule $f(a, b) = -(e^{-a} + e^{-b}) + 2$, with initial reserves $a_0 = b_0 = 1$. The sell offer is an offer to sell 100 units of $A$ in exchange for $B$, with a minimum price of at least $\frac{1}{2} B$ per $A$.

If the final batch exchange rate $p = p_A / p_B$ is strictly greater than $\frac{1}{2}$, then this limit sell offer must sell the entirety of its $A$ to receive at least $100p > 50$ units of $B$. But the CFMM can only provide at most 1 unit of $B$.

On the other hand, if the final batch exchange rate is strictly less than $\frac{1}{2}$, then the limit sell offer will not sell any $A$ but the CFMM would attempt to buy a nonzero amount of $A$.

Thus, at equilibrium, the batch exchange rate must be equal to $\frac{1}{2}$.

Let the amount of $A$ sold to the market by the sell offer be $x$. Clearly, at equilibrium, $0 \leq x \leq 100$.

Furthermore, the spot exchange rate of the CFMM at equilibrium must be equal to $\frac{1}{2}$. The spot exchange rate of this CFMM is

\[
\frac{af}{dx} = e^{-x} \quad \frac{af}{db} = e^{-x} + 1
\]

Thus, at equilibrium, we must have that

\[
p = e^{-(a_0 + x)((b_0 - px))} = e^{-a_0 + x} e^{-x - px} = 1 + e^{-x(1 + p)}
\]

Putting this all together gives

\[
\frac{1}{2} = e^{-x \frac{1}{2}}
\]

which gives $x = \frac{1}{2} \ln(2)$.

Hence, no equilibrium of this market can be rational. \(\square\)

Note that it is not required for this example that the LMSR use the natural logarithm. What is required for this example is that the base of the logarithm $b$ and the limit sell offer’s minimum price $p$ be such that $\log_b(p)$ be irrational.

E Omitted Proofs of §7

E.1 Proof of Proposition 7.1

**Proposition:** A CFMM trading function can be modified to account for trading fees on the CFMM’s net trade in one batch.

**Definition E.1 (Fee Difference Function):** Define $\chi_\epsilon(x) = x$ if $x \leq 0$ and $(1 - \epsilon)x$ for $x \geq 0$.

Let $\hat{x} \in \mathbb{R}^n_{\geq 0}$ denote a CFMM’s initial state. The fee difference function for a fee $\epsilon$ $\chi_\epsilon,\hat{x}(x) = \{\hat{x} + \chi_\epsilon(x - \hat{x})\}_{i \leq n}$.

Observe that $x_i = \chi_\epsilon,\hat{x}(x)_i$ when $x_i \leq \hat{x}_i$ and $(x_i - \hat{x}_i)(1 - \epsilon) + \hat{x}_i$ otherwise. In other words, this function performs the accounting involved in charging a fee $\epsilon$ on the net flow of an asset into the CFMM’s reserves.

**Proof.** The function $f(\chi_\epsilon,\hat{x}(\cdot))$ charges $\epsilon$ fee on every asset that in net enters the CFMM. Using this function as a CFMM’s trading function in a batch exchange means that the CFMM will trade to a set of reserves $x$ where this function’s spot valuations are equal to the batch valuations.

As such, the reserves of the CFMM after charging a $\epsilon$ trading fee (if collected fees are not automatically re-deposited into the CFMM reserves), $\chi_\epsilon,\hat{x}(x)$, are such that the spot valuations of the CFMM are equal to the batch valuations. \(\square\)